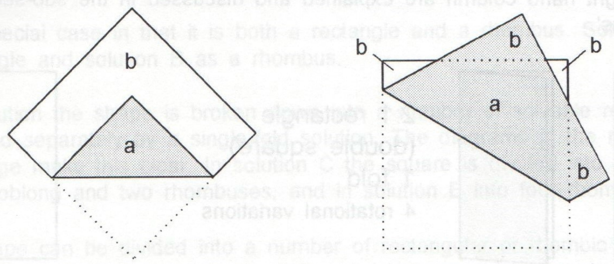
Perfect Mappings

Introduction

One of the results of making any 180 degree fold in a previously unfolded sheet of paper is to map one part of the paper onto another. This has two simple effects. The total area covered by the paper is diminished and part (at least) of the remaining area becomes two layers deep. Where the area covered by the paper is exactly halved, and thus the depth entirely doubled throughout, the mapping can be said to be 'perfect'.



This exploration looks at the perfect mapping solutions that can be obtained for a variety of shapes of paper, first for rectangles (in particular the square and the 2:1 rectangle) then for triangles (in particular the equilateral, silver and bronze triangles) and considers how the solutions for each shape can be related to each other.

It is a reasonable first guess that this exploration is all about symmetry, and the exploration will be presented accordingly, though as the exploration proceeds it will become clear that symmetry is not the only (or indeed perhaps the most important) factor involved.

The special terms and concepts needed to discuss and understand the results of this exploration will be explained as they are met or needed. For now it is sufficient to note that:

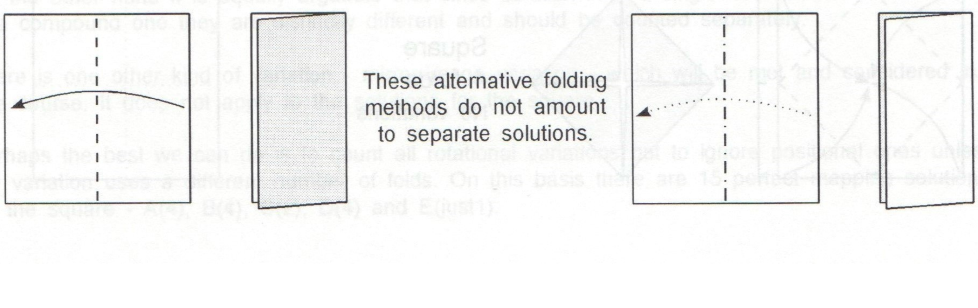
The method by which a perfect mapping is achieved will be called a 'solution'.

Parts of the paper that have already been doubled (eg those marked a on the illustration above) will be referred to as 'solved' regions and those that are still single-layer (marked b) as 'unsolved'.

A region which is capable of solution will be said to be 'solvable' and one which is not capable of solution to be 'unsolvable'.

One of the many interesting questions raised by this exploration is how to define what should (or should not) be counted as a solution. Whether rotational, positional and mirror-image variations count collectively or separately will be discussed when such variations first occur.

For now it is only necessary to note that it is a well established principle of flat origami (origami in which all the folds lie (reasonably) flat) that the outline shape of the paper and the number of layers at each point within that outline are not affected by whether any particular fold is made by moving the paper in an arc towards or away from the folder. Because of this it is clearly inappropriate to distinguish between perfect mapping solutions solely on the basis of the direction of their folds.



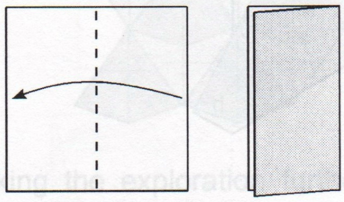
Rectangles

The Square

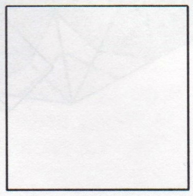
Here are the perfect mapping solutions for the square. The folding diagrams for this exploration have not been given sequence numbers (except right at the end of the exploration where they become more complex). Unnumbered diagrams should always be read from left to right, along and down the page, in the same way as normal text.

The diagrams in the right hand column are explained and discussed in the sub-section 'Single-fold and compound solutions'.

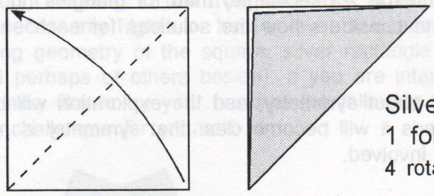
A



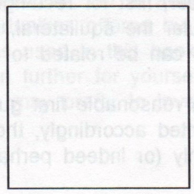
2:1 rectangle
(double square)
1 fold
4 rotational variations



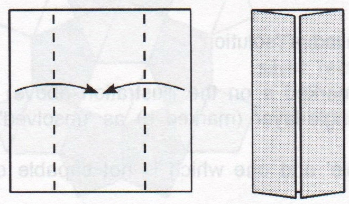
B



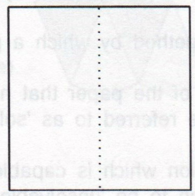
Silver triangle
1 fold
4 rotational variations



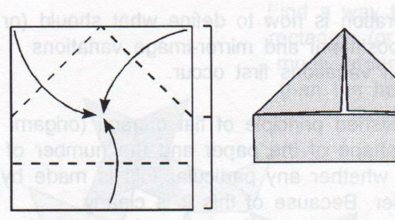
C



2:1 rectangle
(double square)
2 folds
2 rotational variations
Almost infinite positional variations

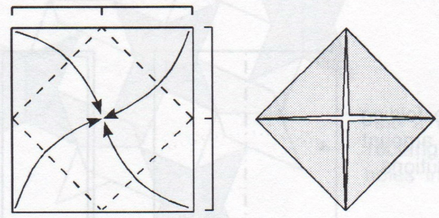


D

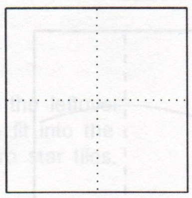


Irregular pentagon
3 folds
4 rotational variations

E



Square
4 folds
No variations



Single-fold and compound solutions

Solutions A and B for the square are achieved in just one fold. Solutions C to E in several. For the purposes of this exploration these two types of solution will be referred to as single-fold and compound solutions respectively.

It will immediately be obvious that single-fold solutions can only exist (and equally must exist) for shapes which possess reflective symmetry. These shapes include all regular polygons, rectangles, isosceles triangles, rhombuses and kite-shapes (both of which can be viewed as a combination of two isosceles triangles joined short edge to short edge).

The square is a special case in that it is both a rectangle and a rhombus. Solution A solves the square as a rectangle and solution B as a rhombus.

In a compound solution the shape is broken down into a number of solvable regions, each of which is then solved separately by a single-fold solution. The diagrams in the right-hand column on the opposite page make this clear. In solution C the square is divided into two oblongs, in solution D into an oblong and two rhombuses, and in solution E into four rhombuses.

The fact that a shape can be divided into a number of rectangular or rhombic etc., regions does not in itself guarantee that a solution based on that division can exist. A region can be solved only if the outside edges of the paper that surround one half of the region (divided along the axis of reflective symmetry) are free to move independently of the rest of the paper.

This idea will is particularly important when we come to consider how to look for and discover solutions that an intuitive investigation may have missed.

Solution or variation?

How many distinctly different perfect mapping solutions are there for the square?

A glance at the opposite page wil show that if solutions are distinguished purely on the basis of outline shape there are just four. However this figure does not take into account the number of folds required or rotational and positional variations.

Solutions A, B, C and D each have rotational variations. This means that, for instance, a similar result to solution A could have been achieved by folding the top edge of the square onto the bottom edge rather than the right onto the left, etc. There is a case for considering rotational variations as one and the same solution (if you were handed these variations to examine you could not distinguish between them once the original orientation had been lost) but, on the other hand, counting each rotational variation as a separate solution gives proper weight to the effect of symmetry when comparing the number of solutions that can be found for different shapes.

Solution C has positional variations. This means that the edges of the folds can meet at any point or position across the square. All the shapes examined in this exploration have at least one solution which has almost infinite positional variations in this way. Therefore it seems unnecessary to count them separately for purposes of comparison, even if we could.

It is arguable that solution A is a positional variation of solution C. This is easy to see If you imagine that the square is rolled into a tube with the edges taped together. Squashing the tube at various points will produce all the possible positional variations of solution C including solution A. On the other hand it is equally arguable that since solution A is a single-fold solution and solution C a compound one they are distinctly different and should be counted separately.

There is one other kind of variation - mirror-image variation - which will be met and considered in due course. It does not apply to the solutions for the square.

Perhaps the best we can do is to count all rotational variations but to ignore positional ones unless the variation uses a different number of folds. On this basis there are 15 perfect mapping solutions for the square - A(4), B(4), C(2), D(4) and E(just1).

Looking for less obvious solutions

This section shows how a process of analytical trial and error can be used to help discover (or in this case to rule out the existence of) less obvious perfect mapping solutions for a particular shape. It is important to understand that while analysis of this kind can show that it is unlikely that any further perfect mapping solutions exist, this does not amount to proof in a mathematical sense, either of the whole or of any of the assertions made along the way. This analysis only applies to the square but the principles explained here can easily be adapted to apply to other shapes.

It is a reasonable assumption that if any further solutions exist they must be compound ones. Therefore we will try to consider all possible ways in which the square can be broken down into potentially solvable regions and then explore whether any of these regions are in fact solvable in context.

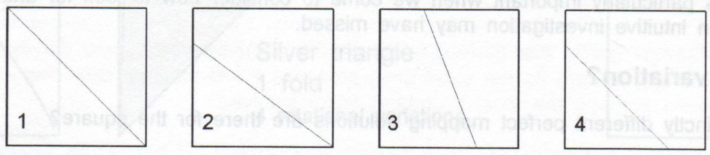
By looking at the square we can see that all the possible first folds must fall into one of only four categories:

Category 1. Those that form a crease running between opposite corners.

Category 2. Those that form a crease running between one corner and an opposite edge.

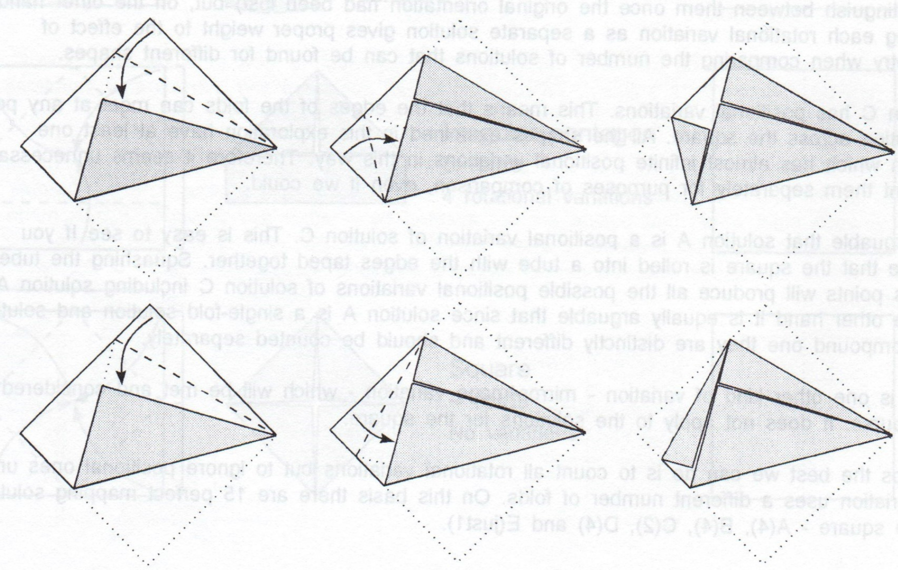
Category 3. Those that form a crease running through two opposite edges.

Category 4. Those that form a crease running through two adjacent edges.

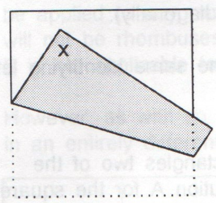


Category 1. We already know that this leads to solution B.

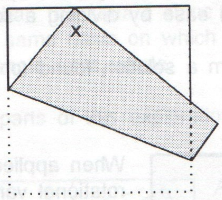
Category 2. All possible positions of this crease solve a kite-shaped region and leave an irregular unsolved region which can be broken down into a second kite-shaped region and right-angle triangle. Only the two shorter edges of this triangle are free. Folding either of them inwards solves a third kite-shaped region but leaves a smaller right-angle triangle. Only one edge of this triangle is free so no further folds are possible.



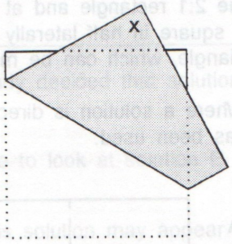
Category 3. In the case of folds that form a crease running through opposite edges, we have already seen that the fold will always result in a solution if the edges are aligned (solutions A and C). If the edges are not aligned a first fold of this kind will always leave either two, three or four unsolved regions. At least one of these will always be a right-angle triangle.



If corner x does not reach the top edge two separate regions are left unsolved. One of these regions is always a right-angle triangle.



If corner x is folded exactly onto the top edge three separate regions are left unsolved. All these regions are always right-angle triangles.

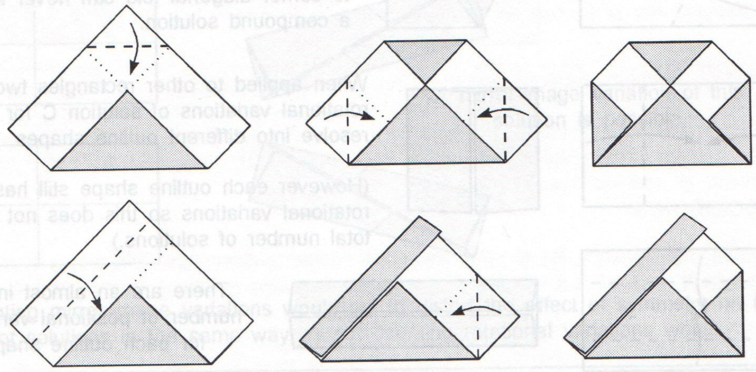


If corner x is folded beyond the top edge four separate regions are left unsolved. All these regions are always right angle triangles.

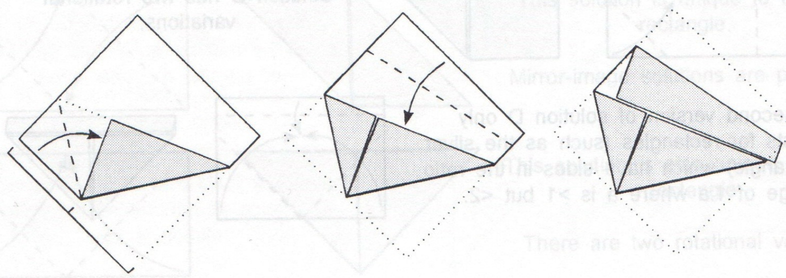
In every case, only the two shorter edges of the triangle(s) are free. We have already seen that such regions cannot be solved.

Category 4. The most symmetrical way to make a fold that forms a crease running through two adjacent edges is to lay the corner between the edges directly onto the diagonal that bisects them. At first sight this approach looks hopeful, since the unsolved region can be sub-divided into rectangles in several ways, but it quickly becomes clear that a solution is only possible if the corner is folded onto the diagonal exactly at the centre of symmetry of the square (which leads to solutions D and E, but no others).

At all other positions, solution of one of the regions into which it is possible to divide the unsolved part of the paper prevents solution of at least one of those that remain unsolved.



If the corner is laid onto a point away from the diagonal the shape of the unsolved regions becomes irregular. These irregular regions are never completely solvable for the reasons already given when looking at categories 2 and 3.

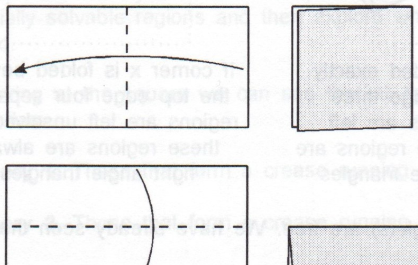


The 2:1 rectangle

This section of the exploration looks at how the solutions obtained for the square can be applied to the 2:1 rectangle and at what happens when they are. This rectangle can easily be made by cutting a square in half laterally (which makes it interesting to compare the solutions to those for the silver triangle, which can be made with equal ease by dividing a square in half diagonally).

Where a solution is directly derived from a solution found for the square the same identifying letter has been used.

A

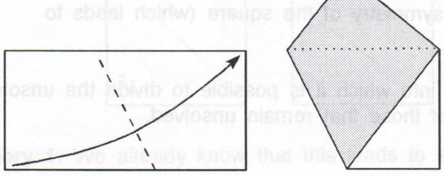


When applied to other rectangles two of the rotational variations of solution A for the square resolve into different outline shapes.

(However each outline shape still has two rotational variations so this does not affect the total number of solutions.)

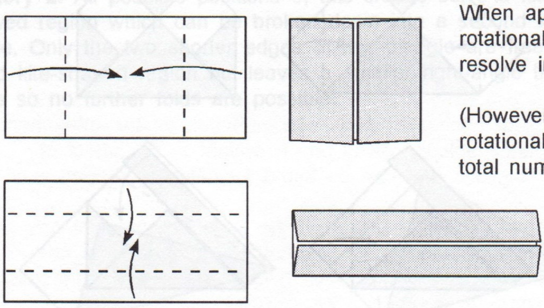
B

Folding corner to corner diagonally does not yield a single-fold solution for any rectangle other than the square.



In addition, since the regions that remain unsolved after the diagonal fold has been made are always right-angle triangles, and neither hypoteneuse is ever free, a corner to corner diagonal fold can never lead to a compound solution.

C

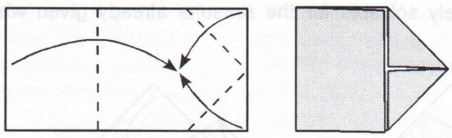


When applied to other rectangles two of the rotational variations of solution C for the square resolve into different outline shapes.

(However each outline shape still has two rotational variations so this does not affect the total number of solutions.)

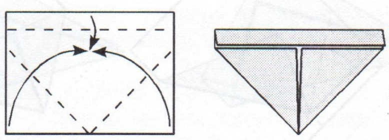
There are an almost infinite number of positional variations for each outline shape.

D

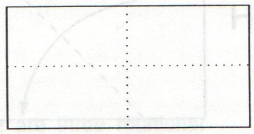


Solution D has two rotational variations.

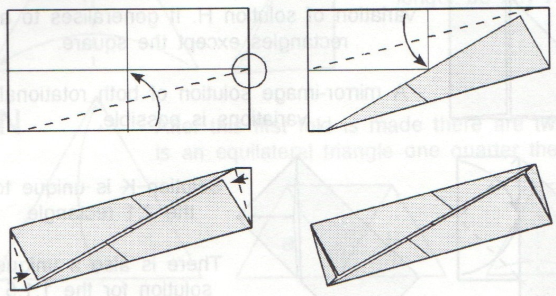
A second version of solution D only exists for rectangles (such as the silver rectangle) which have sides in the ratio range of 1:a where a is >1 but <2.



E Solution E for the square can be seen as a compound version of solution B. The solution works by dividing the square (which we have already noted is both a rectangle and a rhombus) into four regions (each of which is a rhombus of similar shape to the whole). Looked at in this way it is clearly impossible that solution E could be applied to any rectangle other than the square since the regions will not be rhombuses. This is the same basis on which we have already decided that solution B is not applicable to the 2:1 rectangle as a whole.



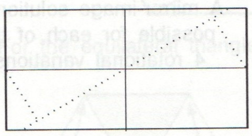
However, as with so many of the parts of this exploration, it is possible to look at solution E in an entirely different way

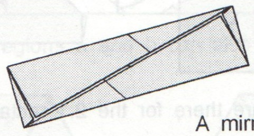


At first sight this solution may appear to have no similarity to solution E for the square, but a consideration of the way in which the folds are located should quickly convince you that this solution generalises to all rectangles.

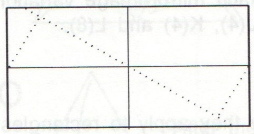
On this premise solution E of the square can be seen as a special case produced by applying the same fundamental rules.

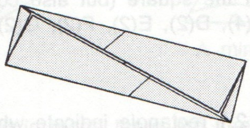
Note that in order to locate the folds the rectangle must first be folded in half both ways to establish the centre of symmetry and the centre of each short edge. The first crease starts at the centre of the right hand short edge (marked with a circle) and is made in such a way that the edge of the rectangle folds over to pass directly through the centre of symmetry. Accurate folding is required. The second crease is easier to make because there is a convenient edge to align it with. The remaining unsolved regions are both kite-shaped and easily solved.





A mirror-image variation of this solution is possible.

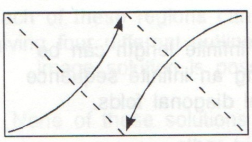


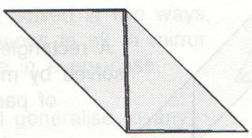


Not counting mirror-image variations would act to distort the effect of symmetry on the number of solutions in the same way as not counting rotational variations would.

Several other compound solutions for the 2:1 rectangle can be discovered. These rely on the solutions for the square but are not directly analogous to them.

F

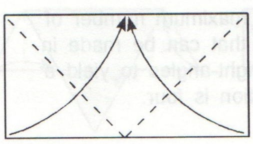


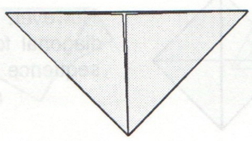


This solution is unique to the 2:1 rectangle.

Mirror-image solutions are possible.

G

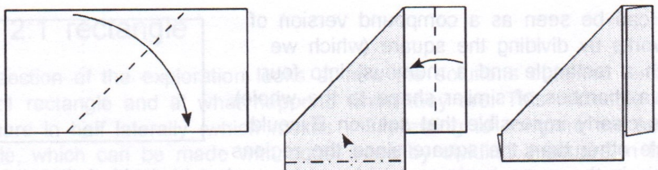




This solution is also unique to the 2:1 rectangle.

There are two rotational variations.

H

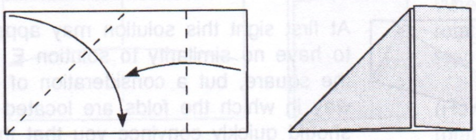


This solution generalises to all rectangles other than the square.

4 rotational variations are possible.

The first fold can be made at any point provided it is made at an angle of 45 degrees to the top and bottom edges, giving almost infinite positional variations.

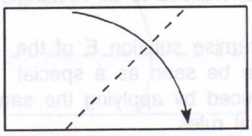
J



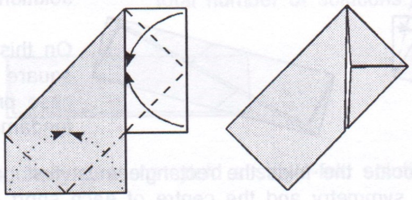
Solution J can be viewed as a positional variation of solution H. It generalises to all rectangles except the square.

A mirror-image solution of both rotational variations is possible.

K



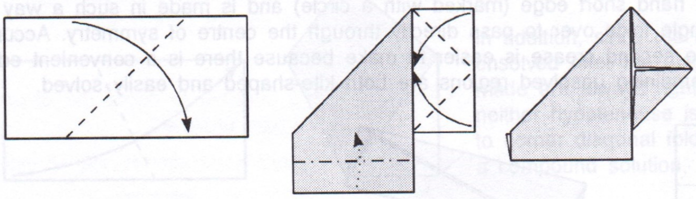
4 rotational variations are possible.



Solution K is unique to the 2:1 rectangle.

There is also a unique solution for the 1:1.5 rectangle.

L



Solution L generalises to all rectangles of 1:a where a is >1.5

A mirror-image solution is possible for each of the 4 rotational variations.

How many perfect mapping solutions are there for the 2:1 rectangle?

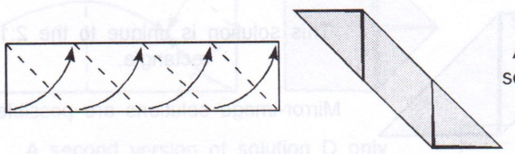
Using the same criteria as we used for the square (but also counting each mirror-image variation separately) the answer is 36 - A(4), C(4), D(2), E(2), F(2), G(2), H(4), J(4), K(4) and L(8).

Shorter rectangles

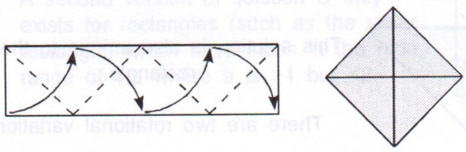
The notes beside the solutions for the 2:1 rectangle indicate where how they apply to rectangles of proportion 1:a where a is >1 but <2. No other perfect mapping solutions unique to any such rectangle are known.

Longer rectangles

Rectangles of proportion 1:a where a is >2 can be solved by the generalised solutions given above or by adapting solutions F and G. Two other interesting results deserve mention.



A rectangle of infinite length can be solved by making an infinite sequence of parallel diagonal folds.



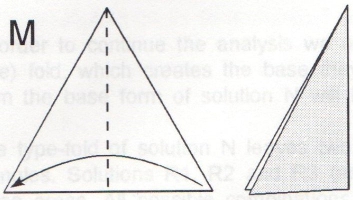
However, the maximum number of diagonal folds that can be made in sequence at right-angles to yield a solution is four.

Triangles

The Equilateral Triangle

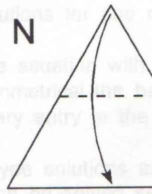
Because the equilateral triangle has three axes of reflective symmetry there are three rotational variations of each solution shown, making a total of 39 solutions in all - M(3), N(12), O(24).

M

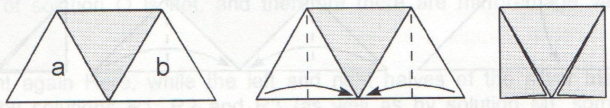


This solution generalises to all other isosceles triangles (although there will no longer be any rotational variations).

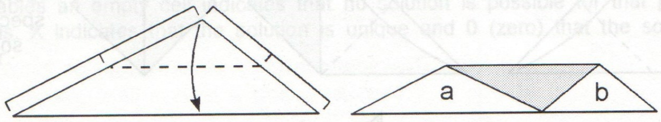
N



After this first fold is made there are two unsolved regions each of which is an equilateral triangle one quarter the area of the starting shape.



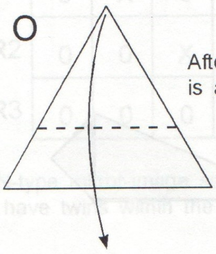
This solution generalises to all other triangles provided the fold is made so that the corner between the two shorter edges of the triangle folds onto the longest edge to form a crease parallel to that edge, since regions a and b will always be isosceles triangles.



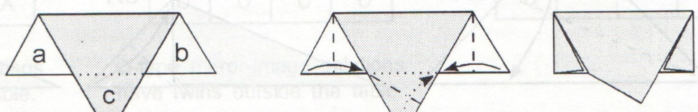
For the equilateral triangle only, regions a and b can also be solved in the following ways.

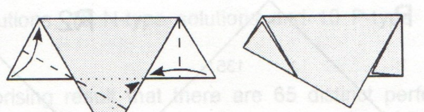
A mirror-image solution is possible.

O



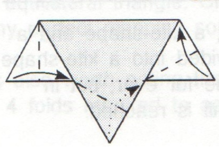
After the first fold is made there are three unsolved regions each of which is an equilateral triangle one ninth the area of the starting shape.

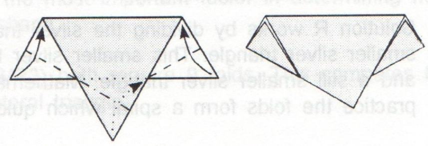




Each of these regions can be solved in two ways, giving four different outline shapes in all. A mirror image solution is possible in every case.

None of these solutions will generalise to any other triangle.

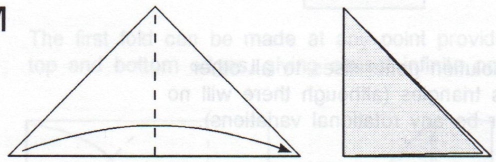




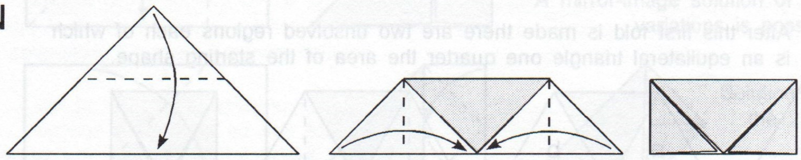
The Silver Triangle

The silver triangle can also be described as the right-angle isosceles triangle and can be made by dividing a square in half diagonally. At first sight the silver triangle is not a promising shape for exploration. It lacks the three-fold rotational symmetry of the equilateral triangle and therefore should allow far fewer solutions.

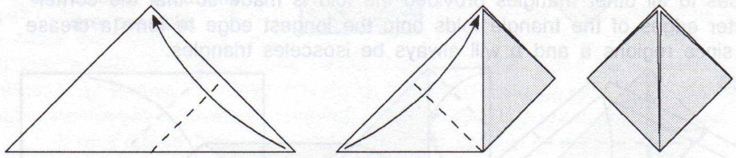
M



N

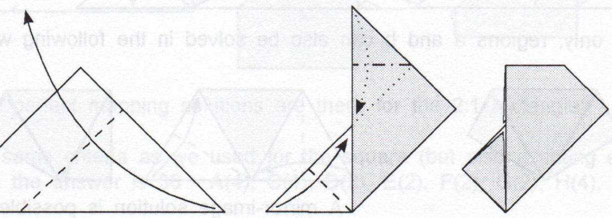


P



Solution P can be seen as a special case of solution Q.

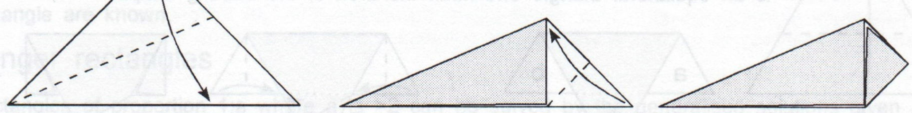
Q

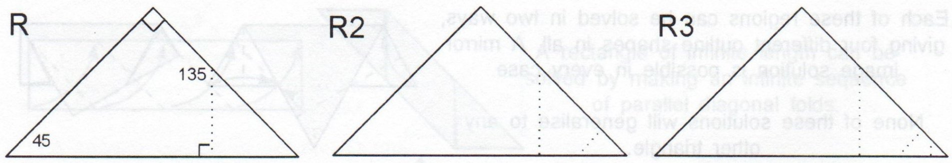


There are almost infinite positional variations of the first fold.

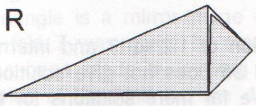
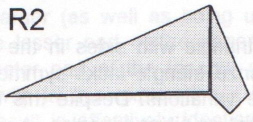
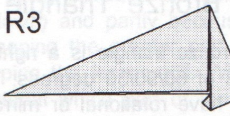
Mirror-image solutions are also possible.

R





Solution R works by dividing the silver triangle into two solvable regions, a kite-shape and a smaller silver triangle. This smaller silver triangle can then in turn be divided into a kite-shape and a still smaller silver triangle. Mathematically this process can continue for ever, but in practice the folds form a spiral which quickly overlaps itself so that a limit is reached.

R

R2

R3


Since a mirror-image solution is possible in each case there are six solutions in all.

In order to continue the analysis we need to begin to classify solutions by reference to the first (or type) fold, which creates the base they are developed from. Thus, for instance, solutions developed from the base form of solution N will be referred to as N-type solutions.

The type-fold of solution N leaves two regions unsolved, both of which are themselves silver triangles. Solutions R1, R2 and R3 (as well as solution M) can be used to solve one or both of these areas. All possible combinations (shown in tabular form below) lead to solutions since the solutions for one region do not interfere with the solutions for the other.

The situation with type-fold Q is essentially similar, except that while the basic form of solution N is symmetrical the basic form of solution Q is not, and therefore there are mirror-image variations for every entry in the table.

P-type solutions are different again Here, while the left and right halves of the silver triangle can each be solved separately by solutions R1, R2 and R3 (as well as by solution M), some of the combinations (R2 with R2, R2 with R3, and R3 with R3) interfere with each other (create more than two layers in places) and do not result in perfect mapping solutions.

In these solution tables an empty cell indicates that no solution is possible for that particular combination of folds. X indicates that the solution is unique and 0 (zero) that the solution is one of a mirror-image pair.

N-type solutions

	M	R1	R2	R3
M	X	0	0	0
R1	0	X	0	0
R2	0	0	X	0
R3	0	0	0	X

N-type mirror-image solutions have twins within the table.

Q-type solutions

	M	R1	R2	R3
M	0	0	0	0
R1	0	0	0	0
R2	0	0	0	0
R3	0	0	0	0

Q-type mirror-image solutions have twins outside the table.

P-type solutions

	M	R1	R2	R3
M	X	0	0	0
R1	0	X	0	
R2	0	0		
R3	0			

Counting mirror-image solutions there are 32 Q-type solutions, 16 N-type solutions and 10 P-type solutions.

Adding all these numbers together we arrive at the surprising result that there are 65 distinct perfect mapping solutions for the silver triangle - M(1), N(16), P(10), Q(32) and R(6) - as opposed to 39 for the equilateral triangle. Clearly symmetry is not after all the most important factor in determining how many perfect mapping solutions exist for any particular shape.

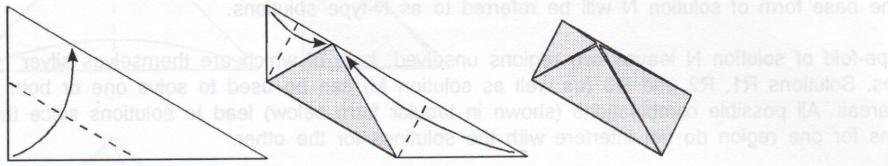
It is also worth noting that solutions N(R3/R3) and Q(R3/R3) both require 9 folds. This compares to the 4 folds required to achieve solution O for the equilateral triangle.

The Bronze Triangle

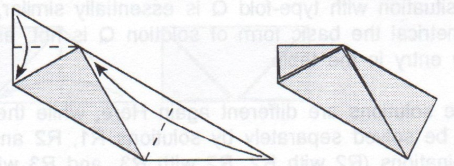
The bronze triangle is a right-angle triangle with sides in the proportion of 1:2:sqrt3 and internal angles of 60/90/30 degrees. The bronze triangle lacks symmetry and so does not give solutions which have rotational or mirror-image variations. Despite this there are far more solutions for the bronze triangle than for the equilateral or silver triangles. An infinity of solutions, in fact.

Where a solution is directly derived from a solution found for the equilateral or silver triangles the same identifying letter has been used.

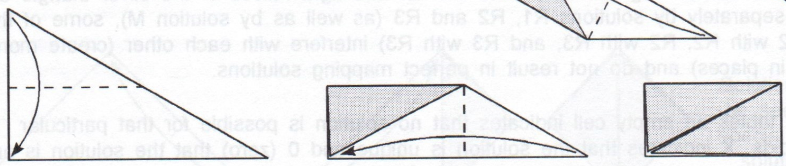
N



Since one of the two regions left unsolved by the type-fold is an equilateral triangle a second version of each N-type solution is possible.

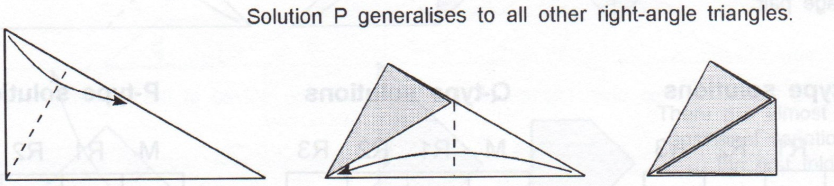


P

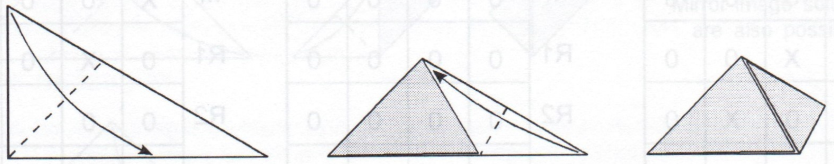


Solution P generalises to all other right-angle triangles.

S



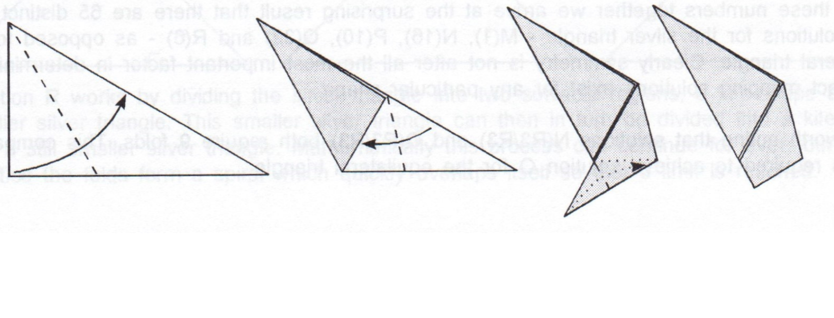
T



In all the solutions above, the region (or in the case of solution N one of the regions) that remains unsolved after the type-fold has been made is a 30/120/30 isosceles triangle. This unsolved triangle is then solved by a fold equivalent to solution M of the silver triangle. If we wanted to describe, for instance, solution S, in a more complete way we could refer to it as Sm, meaning that the unsolved region left by the type-fold S was subsequently solved by a fold equivalent to solution M.

In solution U below the unsolved region is a smaller bronze triangle, which is then partly solved by a fold equivalent to solution T, and finally fully solved by a fold equivalent to solution M. This fold can therefore be designated Utm.

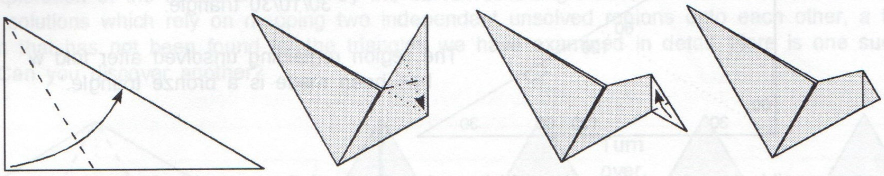
Utm



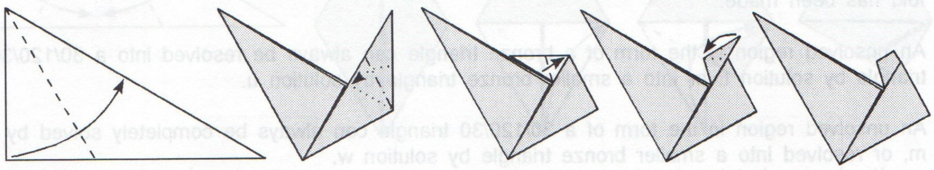
At first sight fold t looks different from the first fold of solution T. This is partly because the smaller bronze triangle is a mirror-image of the larger (as well as being upside down) and partly because whereas fold T was made by moving the lesser part of the paper and keeping the greater part still, fold t has been made by moving the greater part of the triangle and keeping the lesser still. With simple folds of the kind we are dealing with in this exploration it is immaterial which part of the paper moves and which stays still, the result is effectively identical.

Type-fold U can be developed into a sequence of solutions which divide the triangle into similar (but successively smaller) kite-shaped regions, a process that is directly analogous to the development of solutions R1 to R3 for the silver triangle.

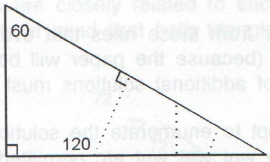
Uuta



Uuuta

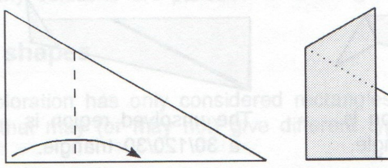


Mathematically, this subdivision into kite-shaped regions can continue ad infinitum, but in practice the folds form a spiral which quickly overlaps itself so that a limit is reached.



There is one final type-fold to be considered. Type-fold V leaves two regions unsolved. One is an equilateral triangle (so there are two versions of every V-type solution) and the other a bronze triangle, which can, of course, be solved by applying partial solutions ta, uta or uuta, giving six variant V-type solutions in all.

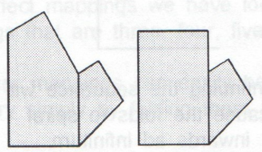
V



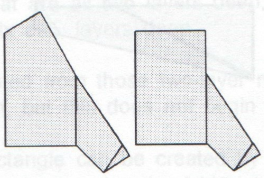
There are almost infinite positional variations of fold V.

Fold S can be seen as a special case of fold V in which only the equilateral triangle remains to be solved.

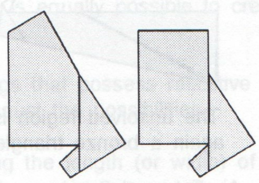
Vta



Vuta

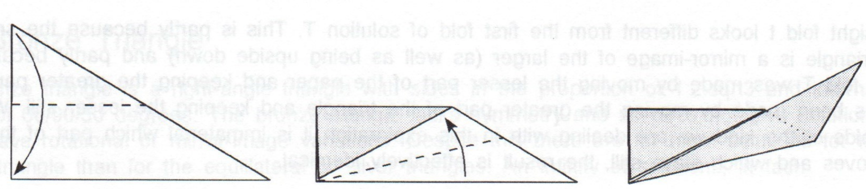


Vuuta

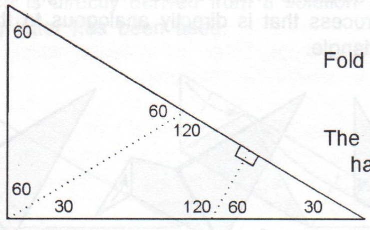


So far we have found 6 type-folds for the bronze triangle, giving 14 solutions in all. There is just one further complication to consider. The 30/120/30 isosceles triangle (encountered at some stage in each of the 14 solutions found so far) can be partly solved to leave an unsolved region in the form of a bronze triangle. We will refer to this fold by the lower case letter w (lower case because it is not a type-fold for the bronze triangle itself).

Pw



Pw is only a partial solution.



Fold w solves a kite-shaped region of the 30/10/30 triangle.

The region remaining unsolved after fold w has been made is a bronze triangle.

It is now possible to set out two rules which explain how solutions can be developed after the type fold has been made:

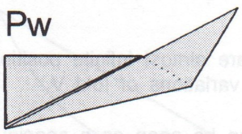
An unsolved region in the form of a bronze triangle can always be resolved into a 30/120/30 triangle by solution t, or into a smaller bronze triangle by solution u.

An unsolved region in the form of a 30/120/30 triangle can always be completely solved by solution m, or resolved into a smaller bronze triangle by solution w.

It is clear from these rules that even though many of the possible sequences will not lead to solutions (because the paper will become more than just two layers thick in places) a considerable number of additional solutions must exist.

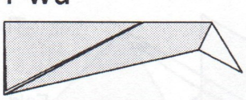
An attempt to enumerate the solutions is not without interest but it soon becomes clear that the sequence wut wut wut etc permitted by the rules spirals inwards without overlapping the regions that have already been mapped onto themselves by previous folds. In practice the unsolved regions soon become too small to fold, but in theory the spiral can be continued for ever, allowing a solution by means of fold m after every fold t in the sequence. Because of this no upper limit can be placed on the number of solutions that are possible.

Pw



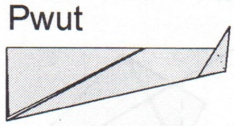
The unsolved region is a bronze triangle.

Pwu



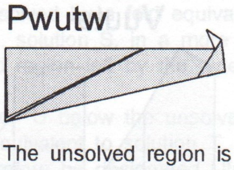
The unsolved region is still a bronze triangle.

Pwut



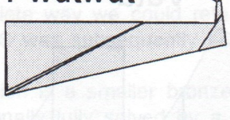
The unsolved region is a 30/120/30 triangle.

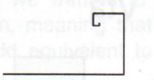
Pwutw



The unsolved region is again a bronze triangle.

Pwutwut





Continuing the sequence will cause the folds to spiral inwards ad infinitum.

There is a sense in which an infinite wut spiral constitutes a solution in itself, since the area of the paper that remains unsolved becomes infinitely small. You may like to consider for yourself the question of how many such solutions are possible for the bronze triangle (which is the same as discovering and counting all the points in the solution sequences at which such spirals can begin).

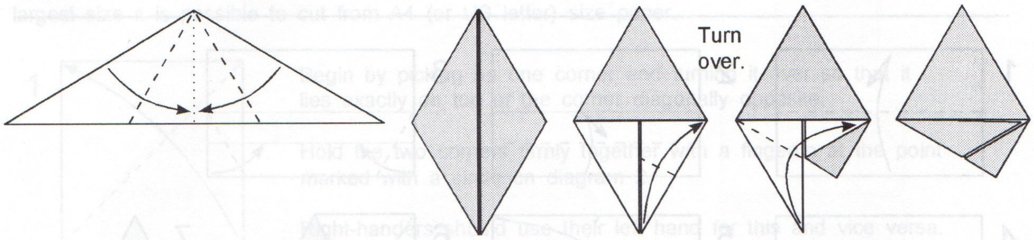
Does the wut spiral solution generalise to apply to all other triangles, just some, or none at all?

Taking the exploration further

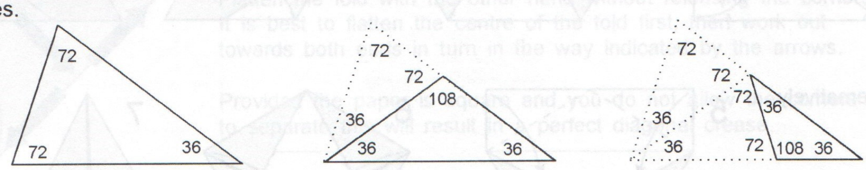
Other triangles

It is quite possible that other triangles may offer results as surprising and interesting as those found for the silver and bronze triangles. Three triangles in particular spring to mind as good candidates for exploration, the 30/120/30 isosceles triangle which figures so largely in the solutions for the bronze triangle (and can be seen as composed of two bronze triangles arranged back to back) and the two golden proportion triangles featured in the exploration of origami tiles and tiling patterns.

A brief exploration of the solutions offered by the 30/120/30 triangle has, for instance, shown that there are solutions which rely on mapping two independent unsolved regions onto each other, a type of solution that has not been found for the triangles we have examined in detail. Here is one such solution. Can you discover another?



The diagrams below show that the two golden proportion triangles are closely related to each other and should provide sufficient hints to kick-start an exploration. Bear in mind that both triangles are isosceles.



It's also worth considering whether solution O of the equilateral triangle might somehow apply.

How many solutions are possible for each golden proportion triangle?

Other shapes

This exploration has only considered rectangles and triangles. There are many other categories of shapes that may (or may not) give different types of solutions.

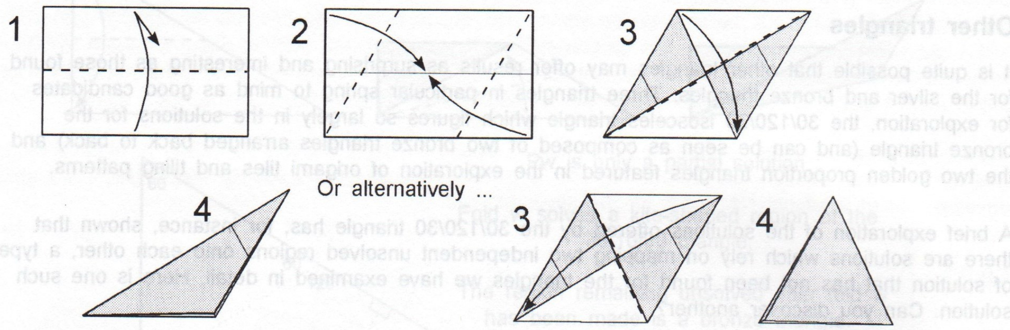
More layers

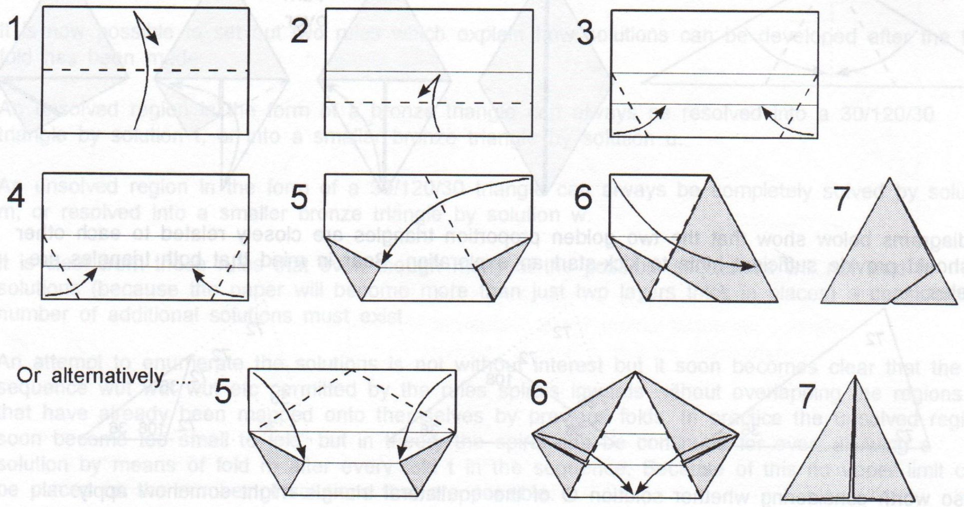
The perfect mappings we have looked at are all two layers deep, but it is equally possible to create mappings that are three, four, five or six etc., layers deep.

Four-layer mappings can easily be created from those two-layer mappings that possess reflective symmetry simply by folding them in half, but this does not begin to exhaust the possibilities.

A basic three-layer mapping for any rectangle can be created by dividing the length (or width) of the rectangle into thirds in either direction and arranging the folds into the form of a flattened Z. (A simple way of achieving this division for the bronze rectangle is given on page 8.) If each of the thirds is then divided into thirds and the folds rearranged in the way shown below a second three-layer mapping is achieved. How far can this sequence be continued?

The bronze rectangle also allows several other interesting three-layer mappings.





At the time this book is being written no exploration of the ways in which the various types of triangle may allow three-layered mappings has been made.

This exploration was previously published in 'Exploring Mathematical Ideas with Origami', David Mitchell, Water Trade, 2001