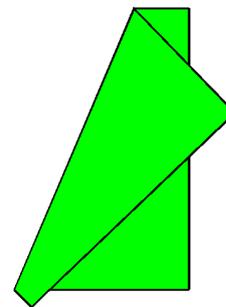
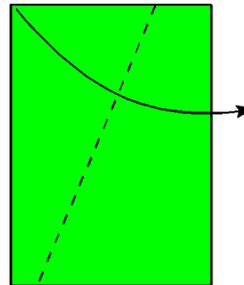


Just One Fold

This pdf looks at the simple mathematical effects of making and flattening a single fold in a sheet of square or oblong paper. The same principles, of course, apply to paper of all shapes.

A fold can be defined as an alteration in direction of the surface of the paper. Flattening a fold produces a crease along the axis of the fold. Forming a crease in this way creates a line of weakness in the paper which can then act as a hinge allowing the fold to be opened out or the direction of the fold to be completely reversed.



Making and flattening a single fold generally alters the flat outline shape of the area covered by the paper and the angles at some of the corners, reduces the perimeter and the area, and results in at least some part of the area covered by the paper becoming two layers deep. Partially opening out the fold results in the form becoming three-dimensional. Completely opening out the fold returns the paper to its original state with the exception that the crease will show up as a line across the paper which divides the paper into two regions.

Each of these effects and the simple mathematical ideas that can be derived from them will be examined in more detail.

Flat outline shapes

With one exception, making and flattening a single fold in a sheet of square or oblong paper, such as A4 or US Letter size paper, alters the outline shape of the paper. Oblongs often yield two alternative solutions. These alternatives are not drawn but you will easily find them for yourself.

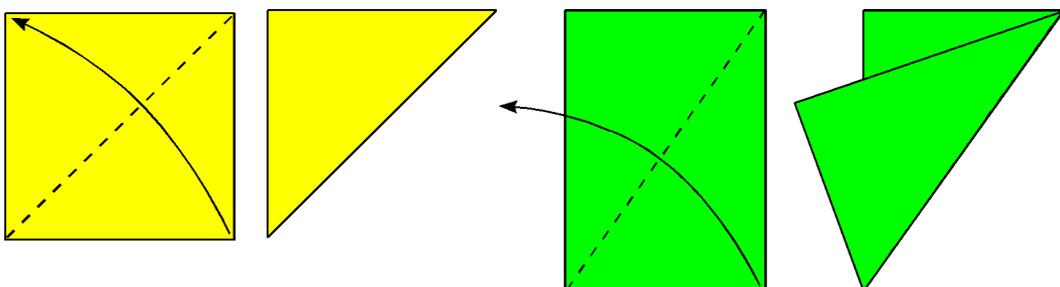
You will notice that it is not possible to create a triangle from an oblong or irregular heptagons or octagons from squares by making and flattening just a single fold. It is also not possible to create a shape with more than nine edges from a square or an oblong in this way.

The exception is that folding a $2:\sqrt{1}$ or silver rectangle in half short edge to short edge produces a half size rectangle of similar shape. Right angle isosceles or silver triangles can similarly be folded in half to obtain a half size triangle of similar shape as can any parallelogram with short and long sides in the proportion of $1:\sqrt{2}$.

It is also a mathematical principle of paperfolding that the outline shape is not dependant on the direction of the fold.

Triangles

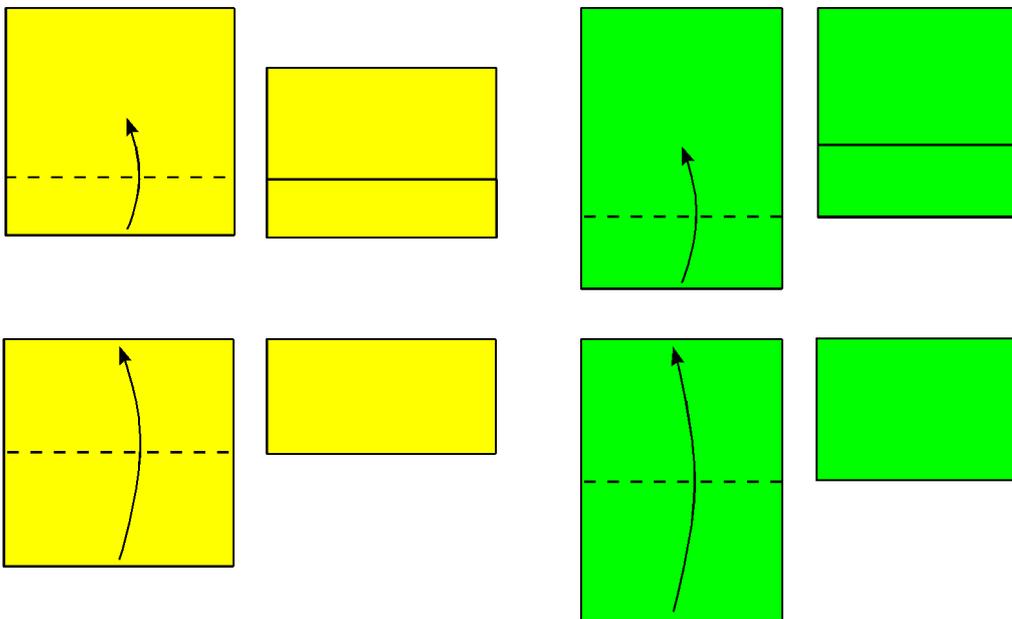
Folding a square in half corner to opposite corner produces a right angle isosceles triangle. It is not possible to create a triangle from an oblong using just one fold. Folding an oblong in half diagonally produces an irregular pentagon instead.



Rectangles

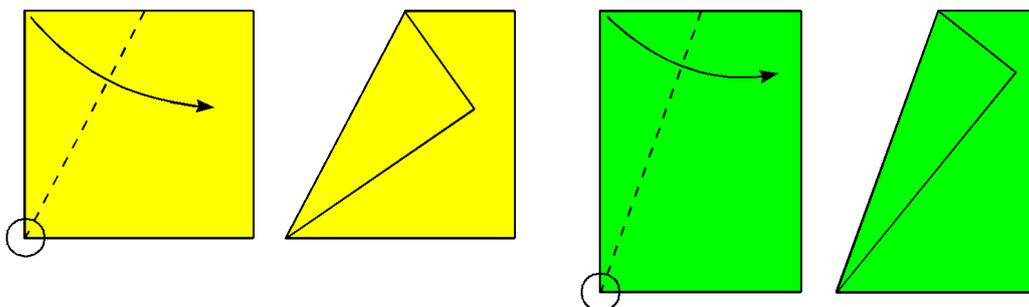
Folding one edge of a square or an oblong inwards in such a way that the two adjacent edges fold onto themselves produces a smaller rectangle.

The easiest way to do this is, of course, to fold edge to opposite edge. The result of doing this to a square is a 2x1 rectangle. The result of doing it to a 1:√2 or silver rectangle, of which A4 paper is a good approximation, is a half-size rectangle of the same proportions as the starting shape..



Quadrilaterals

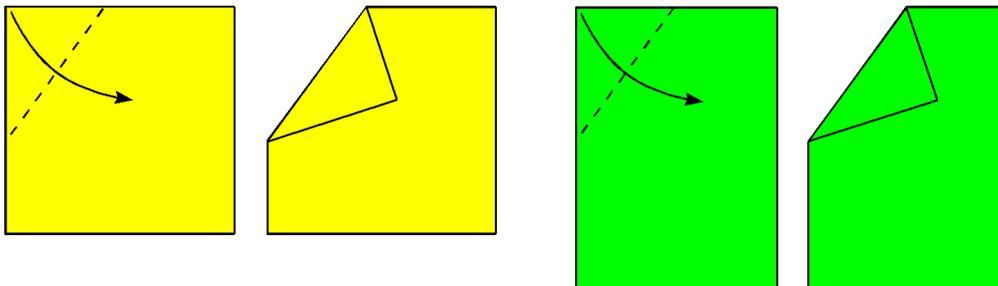
Folding one corner of a square or an oblong inwards so that the crease forms between a corner and an opposite edge produces an irregular quadrilateral. These shapes will tessellate but are not of particular use as tiles because of the difficulty of making them in identical sets.



If however the quadrilateral is created from an oblong by folding one short edge onto an adjacent long edge the result will be a Trapezium tile. Further details about this tile and the interesting way in which it will tessellate can be found in the Origami Tiles and Tiling Patterns section of this site.

Irregular Pentagons

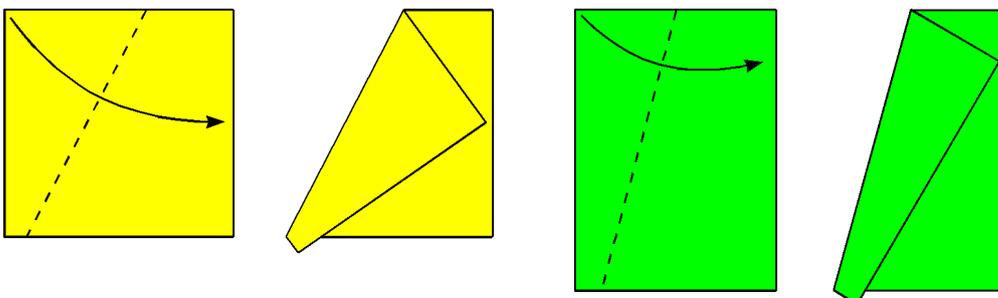
Folding one corner of a square or an oblong inwards so that the crease forms between two adjoining edges and the corner remains within the original area covered by the square or oblong creates an irregular pentagon. These shapes will tessellate but are not of particular use as tiles because of the difficulty of making them in identical sets.



The Cairo Tile, a special kind of irregular pentagon, can be created from a leftover rectangle by making and flattening just a single . Further details about this tile and the interesting way in which it will tessellate can be found in the Origami Tiles and Tiling Patterns section of this site.

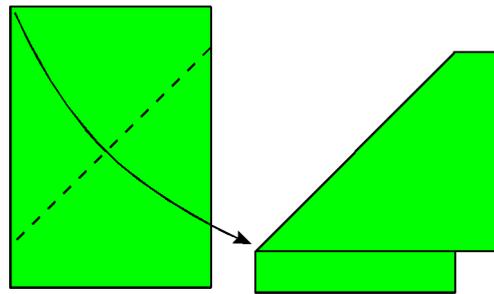
Irregular Hexagons

Folding one corner of a square or an oblong inwards so that the crease forms between two opposite edges but remains within the original area covered by the square produces an irregular hexagon.



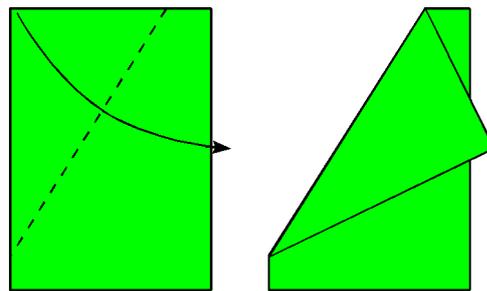
Irregular Heptagons

It is not possible to create a heptagon from a square using just one fold. However, folding an oblong along a central diagonal crease in the way shown below creates an irregular heptagon.



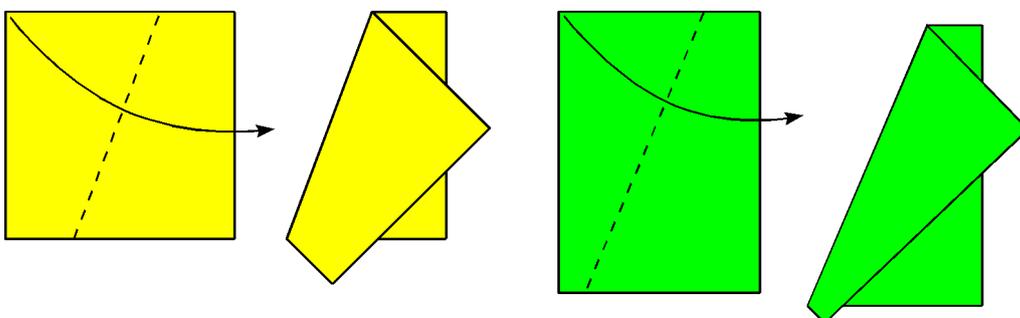
Irregular Octagons

It is not possible to create an octagon from a square using just one fold. However, folding an oblong in the way shown below creates an irregular heptagon.



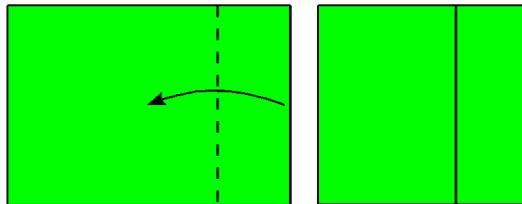
Irregular Nonagons

Folding one corner of a square or an oblong so that the crease forms between two opposite edges but the moving corner ends up outside the original area covered by the square or oblong creates an irregular nonagon.

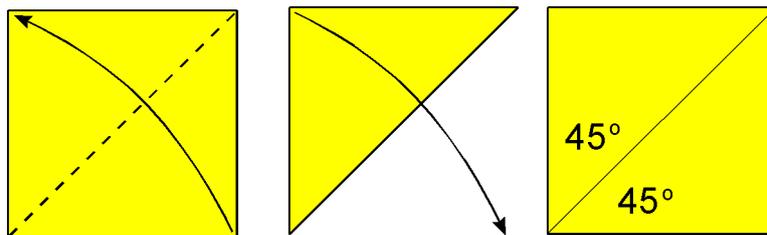


Angle

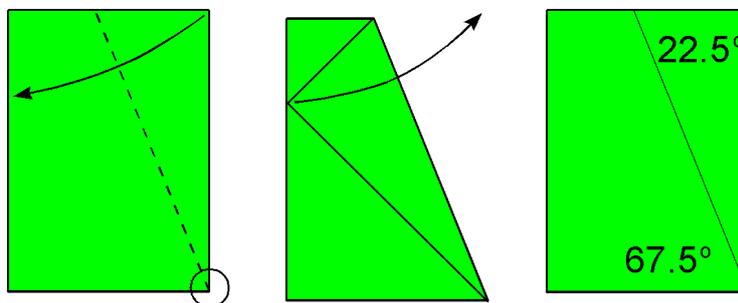
With one exception, making and flattening a fold in a sheet of paper alters the internal angles of the shape. This exception is, of course, found when folding one edge of a rectangle inwards so that it remains parallel to the opposite edge and the outline shape remains a rectangle.



Folding an edge onto an adjacent edge bisects the angle between them. The internal angles of the various regions of a shape or a crease pattern can often be worked out using angle bisection as a starting point.



It is also possible to accurately divide the corner of an oblong, although not a square, by folding one corner onto the opposite long edge in such a way that the crease passes through the adjacent corner at the opposite end of a long edge. If this oblong is a 1:√2 or silver rectangle the resulting crease divides that corner into angles of 67.5 and 22.5.



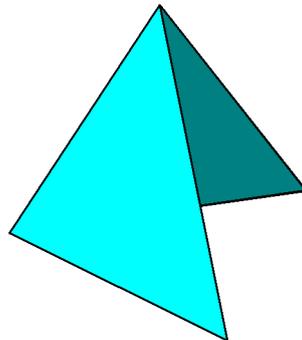
(By using a second crease as a location point it is possible to divide a right angle into angles of 60 and 30 degrees. Other multiple-crease methods of dividing right angles into other geometrically useful angles are known. It is also possible to trisect any given angle by folding paper.)

If a crease made by flattening the paper is lifted upwards using the crease as a hinge the design becomes three-dimensional and it becomes possible to visualise the paper as a solid by imagining the other faces.

The easiest way to see this is to fold a square in half edge to edge, flatten the fold to form a crease, then open up the crease slightly. The fold can be laid on a flat surface, crease upwards, so that it looks like a triangular prism, or a tent.



Similarly, making the fold diagonally will allow you to stand the paper with one point upwards and visualise it as a tetrahedron.



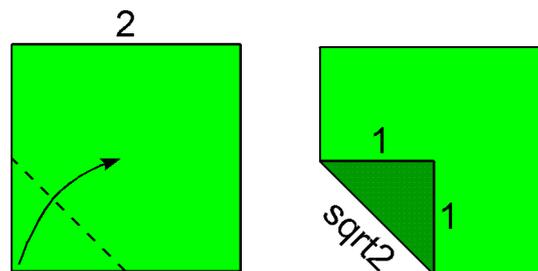
However, the angles of the sides cannot be set with any accuracy so that these folds have limited use mathematically. Volumes, for instance, are much more easily calculated using more robust designs.

Perimeter

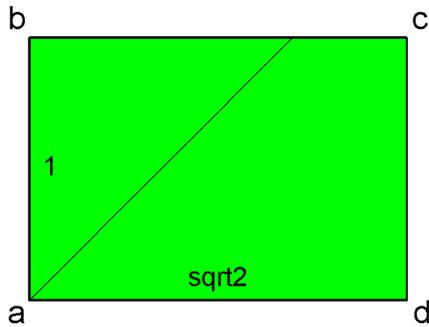
Making and flattening a single fold in a sheet of paper always reduces the perimeter of the paper.

It is easy to see that, for a fold made within the original area of the paper, the remaining perimeter includes the length of the longest edge of the region of the paper folded inwards but does not include the length of all the other edges of that region. The length of the longest edge of a polygon is always less than the sum of the other edges.

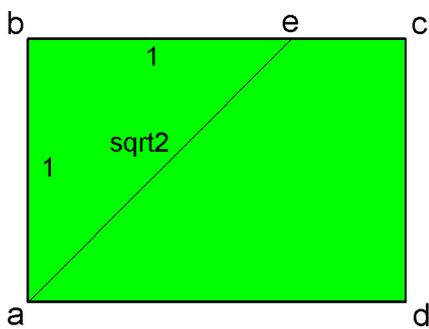
This loss of perimeter is not, of course, easy to quantify if the folds are made randomly. However if, for instance, one corner of a square of side length 2 is folded into the exact centre to create an irregular pentagon the loss of perimeter can easily be quantified as $2 - \sqrt{2}$.



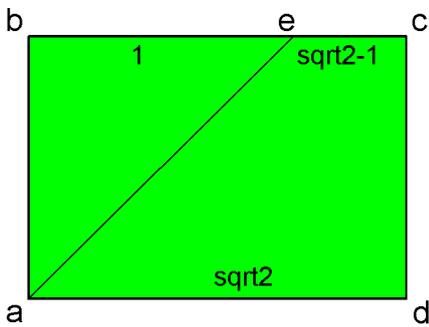
If the single crease is made by bisecting the corner of a $1:\sqrt{2}$ or silver rectangle the resulting quadrilateral - which we have already seen can be used as a Trapezium Tile - can be shown to have a perimeter of $3(\sqrt{2})$ as follows:



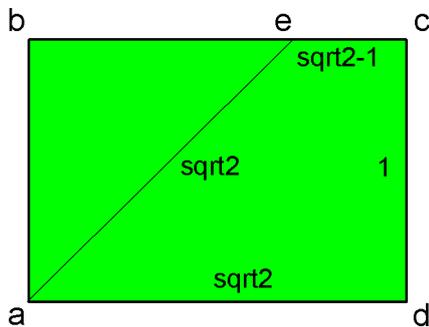
This picture shows the paper after it has been folded and opened out. The starting shape is a silver rectangle with edges in the proportion 1: $\sqrt{2}$.



The area above and to the right of crease be is an isosceles triangle. Length be is therefore the same as length ab . By Pythagoras we know that length ae is $\sqrt{2}$.



Since we know that length bc is $\sqrt{2}$ and be is 1 length ec must be $\sqrt{2}-1$.

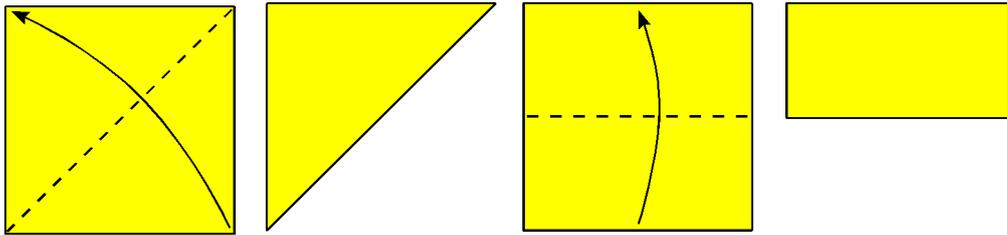


The perimeter of the tile is therefore $3(\sqrt{2})$, three times the length of the long edge of the starting rectangle.

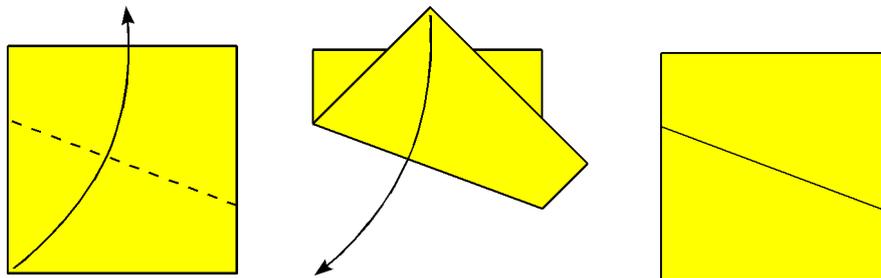
Area

Making and flattening a single fold in a sheet of paper always reduces the area covered by the paper.

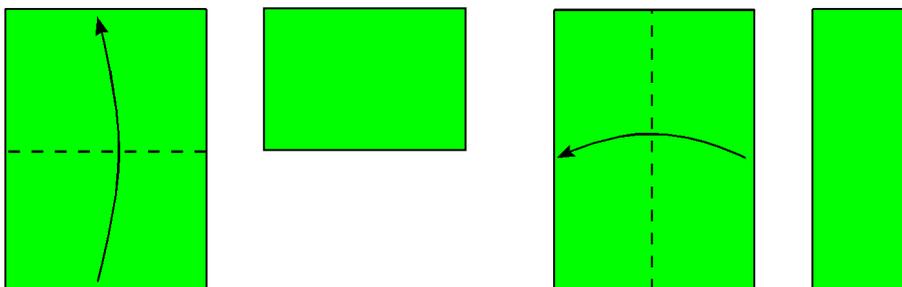
There are two ways of halving the area of a square by making and flattening a single fold.



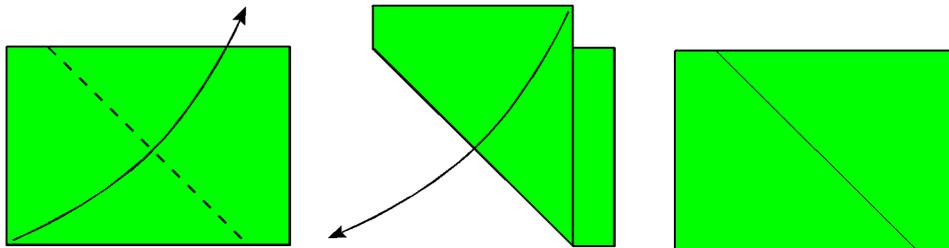
Any other fold which passes through the centre of symmetry of the square will also divide the square into two equal halves, although this will not be obvious until the paper is opened out.



There are two ways of halving the area of an oblong by making and flattening a single fold.



Any other fold which passes through the centre of symmetry of the oblong will also divide the square into two equal halves, although this will not be obvious until the paper is opened out.

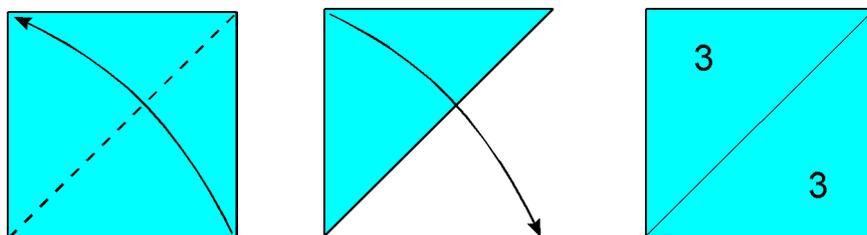


Regions

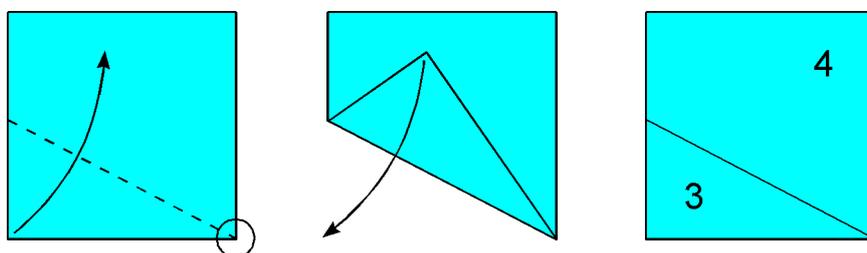
Any crease made in a sheet of paper divides that sheet into two regions. This is best seen if the paper is unfolded so that the crease pattern is visible.

Any single fold made in a rectangle will divide the paper into two regions which are either:

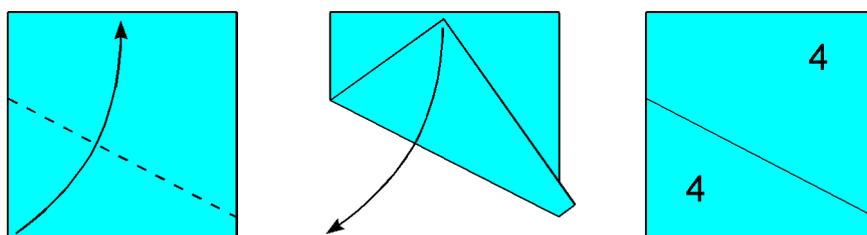
Two triangles:



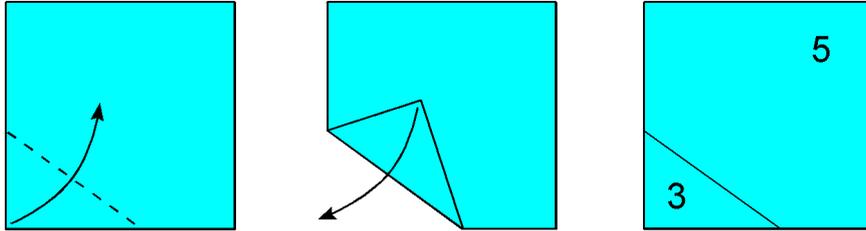
A triangle and a quadrilateral:



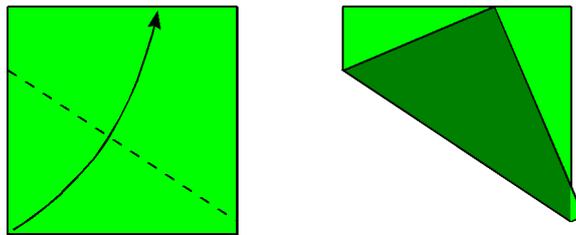
Two quadrilaterals:



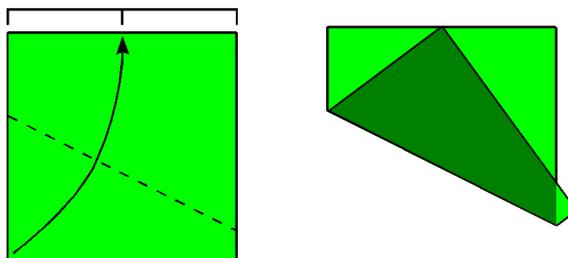
Or a triangle and an irregular pentagon:



Folding a corner of a square onto any point along the opposite edge, except the corners, divides the paper into four regions, one two layers of paper deep and three just a single layer of paper deep. These single layer regions will always be similar right angle triangles. All three will also always be of different sizes.

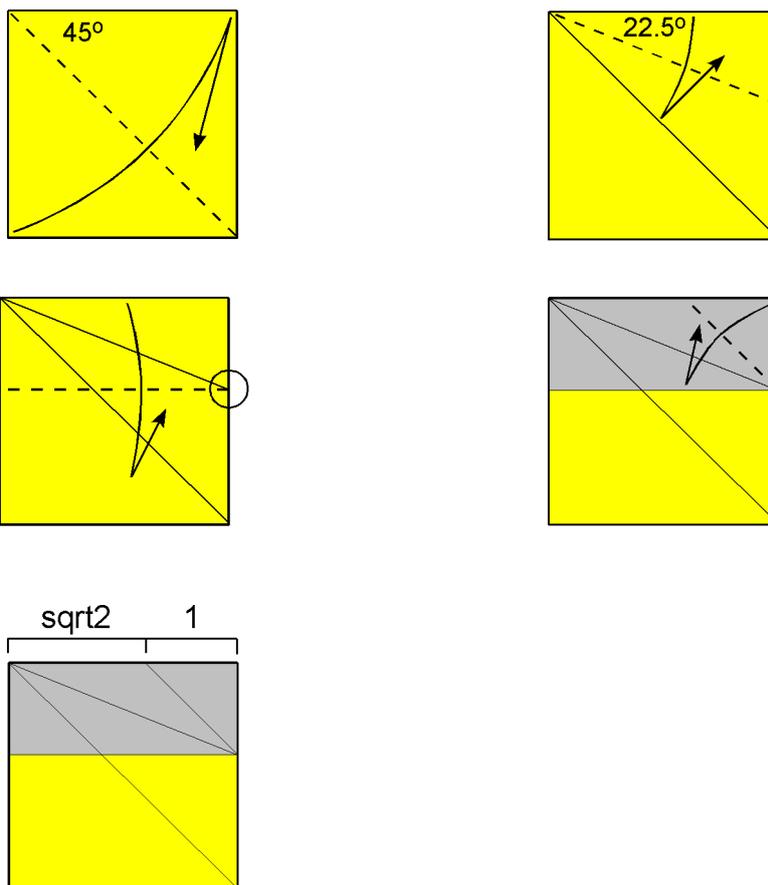


If the corner is folded onto the centre of the opposite edge all the triangles will be of 3:4:5 proportions. Proof of this proposition can be found in Learning Mathematics with Origami by Tung Ken Lam and Sue Pope - Association of Teachers of Mathematics 2016 - ISBN 9781898611950.

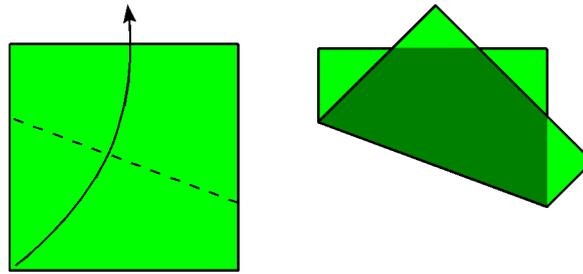


If the corner of a square is folded onto a point that divides the opposite edge into lengths of 1 and $\sqrt{2}$ all the single layer regions will be right angle isosceles or silver triangles. Because of the symmetry of the triangles a fairly accurate result can be achieved by eye alone (which justifies its inclusion as a single fold design). Alternatively the point to which the corner must be folded can be constructed using one of several known methods.

Here is mine, which depends on the knowledge that the diagonals of a $1:1:\sqrt{2}$ or leftover rectangle divide the corners into angles of 22.5° and 67.5° degrees. The grey shaded area is a leftover rectangle. Creasing in the diagonal of a square divides the top edge into the necessary proportions.



Folding a corner of a square to any point above the opposite edge outside the original area of the rectangle divides the paper into five regions, one two layers of paper deep and four just a single layer of paper deep. These four single layer regions will always be similar right angle triangles and all of different sizes, unless the crease passes through the centre of symmetry, in which case they will all be identical.



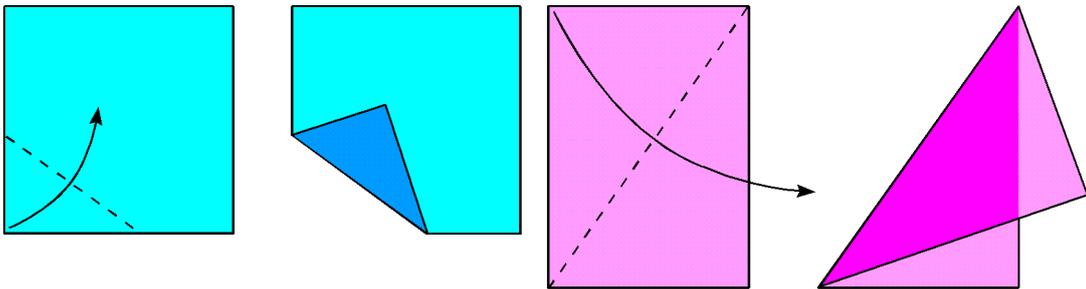
You may like to consider for yourself the difference that starting with an oblong makes to these situations.

Haga's Theorem, which enables the edge of a square to be divided into any given number of equal divisions, also depends on a suitably located first fold of this kind.

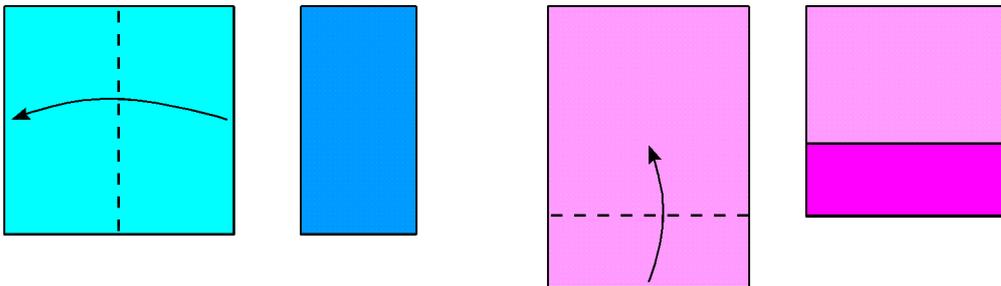
Depth

Making and flattening a single fold in a sheet of paper always results in some part of the area covered by the paper becoming two layers deep. Such double layer regions created by making and flattening a single fold in a square or an oblong may be:

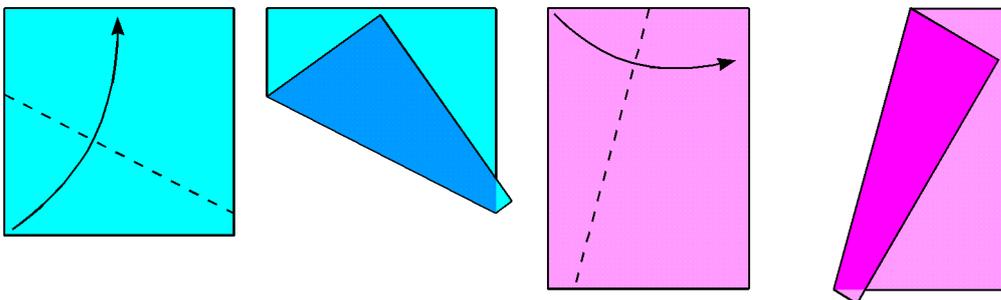
Triangles:



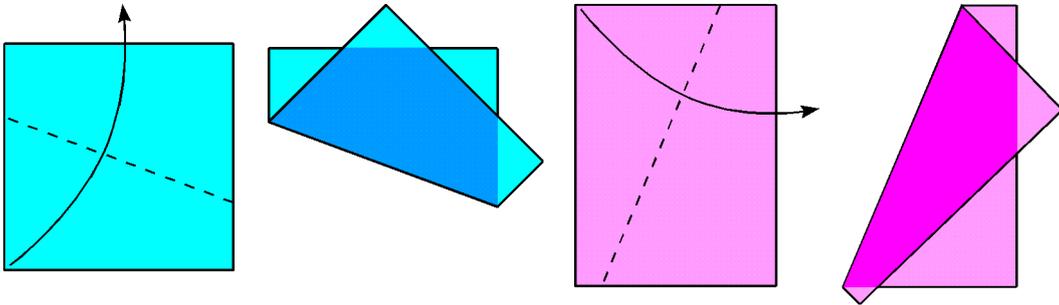
Rectangles:



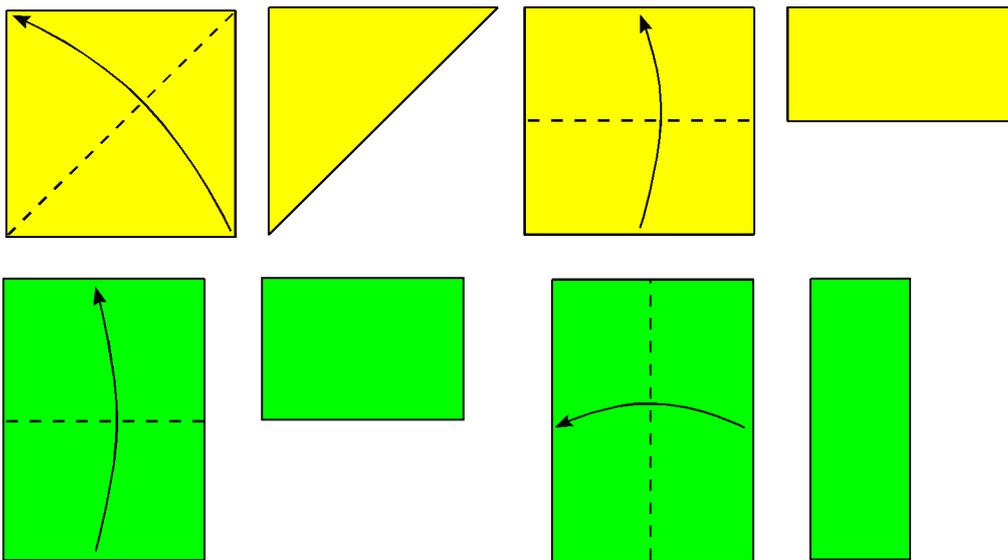
Quadrilaterals:



Or irregular pentagons:



Where the result is two layers deep over the whole area of the folded paper the paper can be said to be perfectly mapped onto itself. Perfect mappings of a square can be obtained by folding it either corner to corner or edge to opposite edge and of an oblong by folding it edge to opposite edge in both alternate directions.



Many other perfect mappings can be obtained using more than just one crease. Investigations of such multiple fold perfect mappings of rectangles, and some particularly interesting triangles, can be found elsewhere on this site.

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